A Constructive Frontier for the Social Golfer Problem

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Abstract

We present constructive lower bounds for the equal-group-size Social Golfer Problem for every ordered parameter pair $2 \leq g, p \leq 50$. For each pair we distinguish a literature or elementary lower bound $L(g,p)$, a constructed number of rounds $C(g,p)$, and a proved upper bound $U(g,p)$. Comparison with the literature located through July 25, 2026 identifies 33 cases in which $C(g,p)$ improves a cited published lower bound. None of these constructions attains the upper bound. A further 616 constructions exceed the available comparison value but are not presented as literature improvements because an appropriate published baseline could not be established. The region $p > g$ is determined by the identity $W(g,p) = 1$; the remaining 1,225 values are printed in a six-part numeric atlas. Complete schedules for all 1,225 nontrivial-region cells, separately indexed schedules for the 33 reported improvements, and the full numerical table are provided as supplementary data. The literature comparison is necessarily dated and may omit results published under different terminology or unavailable in indexed sources.

1 Introduction

The Social Golfer Problem (SGP) asks how many rounds can be scheduled when gp players are partitioned in each round into g groups of size p, with no pair of players meeting twice. Constructions use mutually orthogonal Latin structures, resolvable designs, difference matrices, and related factorizations; computational approaches include constraint programming and symmetry reduction [3, 6, 9].

For each parameter pair, we record the constructed number of rounds, a comparison lower bound with its source, and an upper bound. We report an improvement over the literature only when the comparison lower bound is supported by a cited publication for the same parameters.

Our contributions are as follows.

1. We give constructive values for all 2401 ordered pairs $2 \leq g, p \leq 50$ and compare them with published and elementary bounds.
2. We establish 33 improvements over cited published lower bounds.
3. We identify 616 further cases in which the construction exceeds the available comparison value but the literature is insufficient for a priority claim.
4. We present every nontrivial value in a six-part numeric atlas, with exact values and literature improvements marked directly in the tables.

The companion construction paper develops the parallel-class splitting mechanism and proves five highlighted lower bounds [11]. Here the emphasis is the parameter-wide frontier: the numerical results, their relation to published bounds, and the regions in which the literature comparison remains incomplete.

2 Parameters and comparison conventions

For fixed positive integers g and p, let $W(g,p)$ be the maximum number of rounds in an equal-size SGP schedule on $v = gp$ players. Pair counting gives

$$W(g,p) \leq E(g,p) := \left\lfloor \frac{gp-1}{p-1} \right\rfloor. \tag{1}$$

We consider the ordered pairs $2 \le g, p \le 50$. Rows in every matrix are indexed by g and columns by p.

If $p > g$, then $W(g, p) = 1$. To see this, fix one p-player group in a hypothetical first round. A group in a second round can contain at most one of these players, since two would repeat their pair. The g groups in that round could therefore accommodate at most $g < p$ of the fixed group's players, a contradiction. This determines all 1,176 cells above the diagonal. The atlas prints the remaining 1,225 cells with $p \le g$.

Each cell records three principal quantities:

$L(g, p)$ a lower bound used for comparison, taken from the literature when available and otherwise from an elementary construction;

$C(g, p)$ the number of rounds in the construction reported here; and

$U(g, p)$ a published or elementary upper bound.

The elementary bound $E(g, p)$ in equation (1) is retained separately. Thus $C > L$ is a lower-bound improvement relative to the stated comparison, whereas $C = U$ determines the exact value only when the upper-bound argument applies to the same parameters.

We report a literature improvement only when the comparison value can be tied to a cited publication for the same parameter pair. Where $C > L$ but L is merely elementary, lacks a suitable peer-reviewed source, or remains uncertain, the cell is shown in the atlas but is not included among the 33 literature improvements.

Bold values in the atlas denote improvements over cited lower bounds, and a dagger marks $C = U$. Status glyphs repeat the color information so that the tables do not depend on color alone.

3 Literature, constructions, and verification

3.1 Literature comparison

Lower and upper bounds were collected for the common parameterization in which g is the number of groups and p is the group size. The comparison includes work on the SGP, mutually orthogonal Latin rectangles and squares, resolvable designs, and difference structures. Principal tabulations include Harvey and Winterer [3], Triska and Musliu [9], and the broad 2026 tables of Miller et al. [6]. When the last source labels a value "optimal" without a corresponding maximum proof, we treat it as the best construction reported there rather than as an upper bound.

For the improvements in appendix A, searches included the exact parameters and the corresponding MOLS, MOLR, transversal-design, resolvable-design, and difference-structure terminology. Each constructed value exceeds the strongest published lower bound located in those searches. The parameter- and family-level search covered selected cells, rather than every case with $C > L$; the remaining literature comparisons are incomplete. Section 5 gives the search dates and limitations.

3.2 Construction families

The schedules use several construction families: cyclic partite difference matrices (DM), difference-matrix schedules followed by a factor of the leave (DM+K), resolvable transversal-design trades (RTD-T), mutually orthogonal Latin-square constructions (MOLS), and smaller collections based on nearly Kirkman triple systems, resolvable group-divisible designs, and local search. The companion paper proves the new construction arguments and presents five examples in detail [11]. Classical background on the design constructions is given by Jungnickel [4], Rees and Stinson [8], and Danziger and Stevens [2].

The tables use the abbreviations DM, DM+K, RTD-T, and MOLS just introduced; MOLS+G adds group filling to a MOLS construction, LSC denotes a local-search construction, NKTS a nearly Kirkman triple system, and RGDD4 a resolvable group-divisible design with blocks of size four.

3.3 Schedule verification

A schedule for (g, p, w) was checked for the following conditions:

1. it contains exactly w rounds;
2. each round contains exactly g groups of size p;
3. every round partitions the same set of gp players; and

4. no unordered pair of players occurs together in more than one round.

All 1,225 schedules with $p \le g$ satisfy these conditions and are supplied in the ancillary data. For $p > g$, a single round suffices, as shown in section 2.

4 Results

4.1 The parameter table

The table contains a construction for each of the 2401 ordered parameter pairs. In 1345 cells the construction matches a known exact value, and in 407 cells it matches the comparison lower bound while the corresponding upper bound is larger. Altogether 1354 constructions attain the stated upper bound. Only 178 of these contain more than one round; most of the equalities occur in the elementary region $p > g$.

There are 33 improvements over cited published lower bounds. Their construction methods are distributed as follows:

Construction method	Cells
DM	21
DM+K	6
RTD-T	3
ITD hole completion	1
cyclic development	1
cyclic development + latent-class split	1

Difference-matrix constructions account for most of these results. None of the improved constructions attains the corresponding upper bound.

Nine further exact values occur among the cells whose literature comparison is incomplete:

$$W(31,6) = 37, \quad W(31,16) = 33, \quad W(36,8) = 41,$$
$$W(37,5) = 46, \quad W(38,4) = 50, \quad W(43,4) = 57,$$
$$W(44,4) = 58, \quad W(49,4) = 65, \quad W(49,7) = 57.$$

These equalities follow from the constructions and matching upper bounds; no claim is made about their publication priority.

4.2 Largest improvements

Table 1 lists the ten largest differences $C - L$ among the 33 literature improvements.

Table 1: Largest improvements over cited lower bounds.

g	p	L	C	U	$C-L$	Method
21	7	8	16	24	8	cyclic development + latent-class split
26	7	13	20	30	7	DM+K
18	8	6	11	20	5	DM
22	7	11	16	25	5	DM
10	8	5	9	11	4	DM+K
22	6	14	18	26	4	DM
22	8	11	15	25	4	DM+K
26	8	13	17	29	4	DM
26	9	13	17	29	4	DM+K
12	8	6	9	13	3	DM

The largest difference is 8. The complete list in appendix A gives g, p, L, C, U, the improvement, exactness, method, and cited comparison source. Of these cases, 23 improve entries in the 2026 tables of Miller et al. [6].

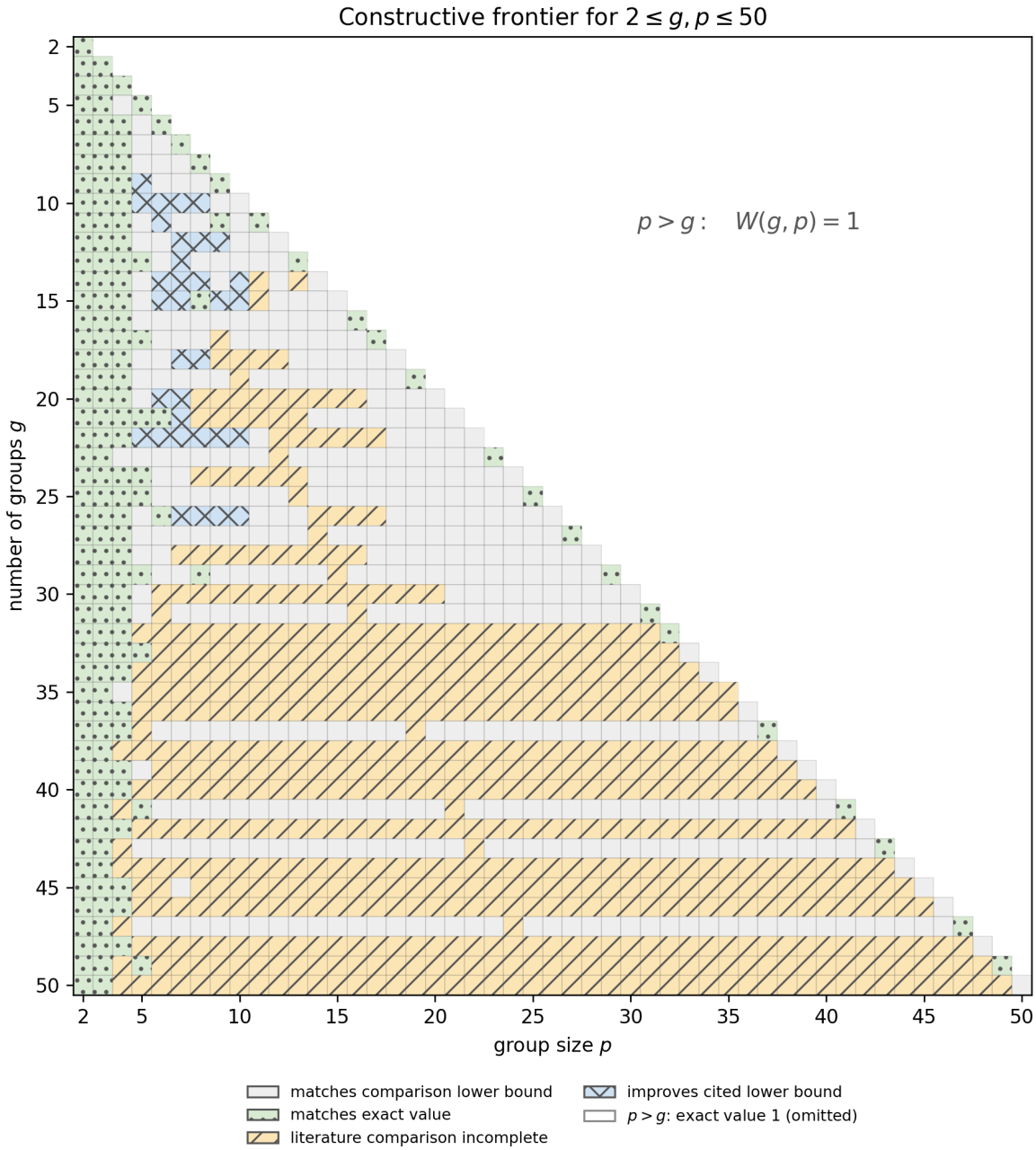


Figure 1: Constructive values for $2 \le g, p \le 50$. The upper triangle is omitted because $W(g, p) = 1$ for $p > g$. Exact numerical values appear in appendix B; colors are repeated by glyphs and typographic markers.

The five lower bounds developed in the companion construction manuscript occur in this dataset [11]. The comparisons from Harvey and Winterer [3] are 14 rounds at $(20, 7)$ and eight rounds at $(21, 7)$ and $(28, 7)$. At $(28, 7)$ the broader literature comparison remains incomplete. At $(14, 13)$, $L = 5$ denotes a five-round construction without a verified published comparison: the relevant entry in Harvey–Winterer's Table 1 was unreadable in the available copy.

5 Limits of the literature comparison

There are 616 cells in which the construction exceeds the available comparison value but which are not included among the 33 literature improvements. The reasons are summarized below.

Reason excluded from the improvement list	Cells
comparison value is elementary	545
no peer-reviewed comparison located	69
published comparison available; broader literature incomplete	1
published construction available; priority unresolved	1

In most of these cases the comparison value is an immediate elementary lower bound rather than a published table entry. Such a construction may be routine, may be new, or may already be known under different design-theoretic terminology. The numerical inequality $C > L$ alone does not distinguish these possibilities.

Published existence results for nearly Kirkman systems, resolvable group-divisible designs, and MOLS already give, respectively, $W(32, 3) = 47$, $W(39, 4) = 51$, and $W(39, 5) \geq 39$ [6, 8]. The order-39 bounds are taken from Miller et al. [6]; the original construction papers were not inspected. These three endpoints are already known. For $(45, 5)$, a corrected $(45, 7, 1)$ difference matrix gives 45 transversal rounds after two point groups are removed; partitioning the five remaining 45-point groups into blocks of size five adds one round [1]. Thus the comparison lower bound is at least 46, while the publication priority of the 55-round construction remains unresolved; that cell is likewise excluded. The same corrected difference matrix develops directly to an $\mathrm{RTD}(7, 45)$, giving the published lower bound $W(45, 7) \geq 45$ and matching the 45-round construction reported here. The interval $45 \leq W(45, 7) \leq 52$ remains open.

The literature cutoff is July 25, 2026, with targeted searches through September 7, 2026. Later publications, unindexed sources, alternative parameter conventions, or corrections to published tables may change some comparisons. The 33 improvements are relative to the cited values located by those dates; the search does not establish first-publication priority. Neither the literature search nor the construction methods are exhaustive, and no conclusion is drawn for parameters outside $2 \leq g, p \leq 50$.

6 Supplementary data

The accompanying supplement contains the directory `anc/`, with the complete numerical table in JSON and CSV, bound provenance, the list of 33 literature improvements, and `frontier-certificates.jsonl.gz`: complete schedules for all 1,225 cells with $p \leq g$. Players are represented by integers, and each schedule is a list of rounds and groups. Running `python3 validate_supplement.py .` inside that directory checks the inventory, duplicated fields, arithmetic, source references, schedule hashes, round partitions, and pair uniqueness.

The validator checks the supplied schedules without rerunning the searches that produced them.

AI assistance disclosure

ChatGPT Pro and the Pi coding-agent harness connected to OpenAI Codex assisted construction exploration, code development, literature searching, data reconciliation, testing, and manuscript drafting. The companion manuscript records the exact model identifiers for the five highlighted constructions. Model identities are not available for every construction in the broader table. The author selected the scope and claims and takes responsibility for the paper.

7 Conclusion

We have given constructive values for the equal-size Social Golfer Problem on all ordered parameter pairs $2 \leq g, p \leq 50$. The comparison yields 33 improved lower bounds; none attains a matching upper bound. Difference matrices

and their leave extensions account for most of the improvements, while transversal-design trades, Latin-square constructions, resolvable designs, and local search supply the remainder.

For the 616 cases with incomplete literature comparisons, the priority questions remain unresolved. Further work includes searching for earlier constructions under alternative terminology, extending the table beyond 50, and constructing schedules for cells with large gaps between the lower and upper bounds.

A Improved lower bounds

Table 2: Constructive improvements over cited published lower bounds.

g	p	L	C	U	$C-L$	Exact	Method	Comparison source
9	5	9	10	11	1	no	RTD-T	[5, 6]
10	5	9	10	12	1	no	ITD hole completion	[6]
10	6	7	9	11	2	no	DM+K	[6, 10]
10	7	7	8	11	1	no	DM	[6, 10]
10	8	5	9	11	4	no	DM+K	[6, 10]
11	6	11	12	13	1	no	RTD-T	[6]
12	7	7	9	13	2	no	DM	[6, 10]
12	8	6	9	13	3	no	DM	[6, 10]
12	9	6	7	13	1	no	DM	[6]
13	7	13	14	15	1	no	RTD-T	[6]
14	6	10	11	16	1	no	DM	[6]
14	7	9	11	16	2	no	DM+K	[6, 10]
14	8	8	9	15	1	no	DM	[6, 10]
14	10	5	6	15	1	no	DM	[6]
15	6	10	12	17	2	no	DM	[6, 10]
15	7	9	10	17	1	no	DM	[6, 10]
15	9	7	8	16	1	no	DM	[6, 10]
15	10	5	7	16	2	no	DM	[6]
18	7	10	13	20	3	no	DM	[6, 10]
18	8	6	11	20	5	no	DM	[6]
20	6	13	16	23	3	no	DM	[6, 10]
20	7	14	15	23	1	no	cyclic development	[3]
21	7	8	16	24	8	no	cyclic development + latent-class split	[3]
22	5	19	20	27	1	no	DM	[6]
22	6	14	18	26	4	no	DM	[6, 10]
22	7	11	16	25	5	no	DM	[7]
22	8	11	15	25	4	no	DM+K	[7]
22	9	11	13	24	2	no	DM	[7]
22	10	11	12	24	1	no	DM	[7]
26	7	13	20	30	7	no	DM+K	[7]
26	8	13	17	29	4	no	DM	[7]
26	9	13	17	29	4	no	DM+K	[7]
26	10	13	14	28	1	no	DM	[7]

B Tables of constructive lower bounds

Table 3: Constructive lower bounds $C(g,p)$ for $p \le g$: $g = 2$–18, $p = 2$–18. Blank cells have $p > g$ and hence $W(g,p) = 1$.

$g\backslash p$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
2	$3^{\dagger}$																
3	$5^{\dagger}$	$4^{\dagger}$															
4	$7^{\dagger}$	$4^{\dagger}$	$5^{\dagger}$														
5	$9^{\dagger}$	$7^{\dagger}$	5°	$6^{\dagger}$													
6	$11^{\dagger}$	$8^{\dagger}$	$7^{\dagger}$	6°	$3^{\dagger}$												
7	$13^{\dagger}$	$10^{\dagger}$	$9^{\dagger}$	7°	7°	$8^{\dagger}$											
8	$15^{\dagger}$	$11^{\dagger}$	$10^{\dagger}$	8°	8°	8°	$9^{\dagger}$										
9	$17^{\dagger}$	$13^{\dagger}$	$11^{\dagger}$	**10**	9°	9°	9°	$10^{\dagger}$									
10	$19^{\dagger}$	$14^{\dagger}$	$13^{\dagger}$	**10**	**9**	**8**	**9**	5°	4°								
11	$21^{\dagger}$	$16^{\dagger}$	$14^{\dagger}$	11°	**12**	11°	11°	$12^{\dagger}$	11°	$12^{\dagger}$							
12	$23^{\dagger}$	$17^{\dagger}$	$15^{\dagger}$	12°	13°	**9**	**9**	**7**	6°	6°	7°						
13	$25^{\dagger}$	$19^{\dagger}$	$17^{\dagger}$	$16^{\dagger}$	13°	**14**	13°	13°	13°	13°	13°	$14^{\dagger}$					
14	$27^{\dagger}$	$20^{\dagger}$	$18^{\dagger}$	14°	**11**	**11**	**9**	7°	**6**	$6^{\triangle}$	5°	$6^{\triangle}$	6°				
15	$29^{\dagger}$	$22^{\dagger}$	$19^{\dagger}$	16°	**12**	**10**	$17^{\dagger}$	**8**	**7**	$7^{\triangle}$	5°	5°	5°	6°			
16	$31^{\dagger}$	$23^{\dagger}$	$21^{\dagger}$	16°	16°	16°	17°	16°	16°	16°	16°	16°	16°	16°	$17^{\dagger}$		
17	$33^{\dagger}$	$25^{\dagger}$	$22^{\dagger}$	$21^{\dagger}$	17°	17°	17°	$18^{\triangle}$	17°	17°	17°	17°	17°	17°	17°	$18^{\dagger}$	
18	$35^{\dagger}$	$26^{\dagger}$	$23^{\dagger}$	18°	19°	**13**	**11**	$11^{\triangle}$	$9^{\triangle}$	$8^{\triangle}$	$7^{\triangle}$	6°	6°	6°	6°	6°	7°

Blank: $p > g$, so $W(g,p) = 1$ (omitted). **Bold**: improves a cited lower bound; †: attains the upper bound; ∘: matches the comparison lower bound; △: literature comparison incomplete.

Table 4: Constructive lower bounds $C(g,p)$ for $p \le g$: $g = 19$–35, $p = 2$–18. Blank cells have $p > g$ and hence $W(g,p) = 1$.

$g\backslash p$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
19	$37^{\dagger}$	$28^{\dagger}$	$25^{\dagger}$	19°	19°	19°	19°	19°	$20^{\triangle}$	19°	19°	19°	19°	19°	19°	19°	19°
20	$39^{\dagger}$	$29^{\dagger}$	$26^{\dagger}$	21°	**16**	**15**	$13^{\triangle}$	$12^{\triangle}$	$11^{\triangle}$	$9^{\triangle}$	$8^{\triangle}$	$7^{\triangle}$	$7^{\triangle}$	$6^{\triangle}$	$6^{\triangle}$	5°	5°
21	$41^{\dagger}$	$31^{\dagger}$	$27^{\dagger}$	$26^{\dagger}$	$25^{\dagger}$	**16**	$13^{\triangle}$	$12^{\triangle}$	$11^{\triangle}$	$10^{\triangle}$	$8^{\triangle}$	$8^{\triangle}$	6°	6°	6°	6°	6°
22	$43^{\dagger}$	$32^{\dagger}$	$29^{\dagger}$	**20**	**18**	**16**	**15**	**13**	**12**	12°	$9^{\triangle}$	$8^{\triangle}$	$7^{\triangle}$	$6^{\triangle}$	$5^{\triangle}$	$5^{\triangle}$	4°
23	$45^{\dagger}$	$34^{\dagger}$	29°	23°	23°	23°	23°	23°	23°	23°	$24^{\triangle}$	23°	23°	23°	23°	23°	23°
24	$47^{\dagger}$	$35^{\dagger}$	$31^{\dagger}$	$29^{\dagger}$	25°	24°	$25^{\triangle}$	$14^{\triangle}$	$13^{\triangle}$	$12^{\triangle}$	$11^{\triangle}$	$9^{\triangle}$	8°	8°	8°	8°	8°
25	$49^{\dagger}$	$37^{\dagger}$	$33^{\dagger}$	$31^{\dagger}$	25°	25°	25°	25°	25°	25°	25°	$26^{\triangle}$	25°	25°	25°	25°	25°
26	$51^{\dagger}$	$38^{\dagger}$	$34^{\dagger}$	26°	$31^{\dagger}$	**20**	**17**	**17**	**14**	13°	13°	14°	$8^{\triangle}$	$7^{\triangle}$	$6^{\triangle}$	$6^{\triangle}$	5°
27	$53^{\dagger}$	$40^{\dagger}$	$35^{\dagger}$	27°	27°	27°	27°	28°	27°	27°	27°	27°	$28^{\triangle}$	27°	27°	27°	27°
28	$55^{\dagger}$	$41^{\dagger}$	$37^{\dagger}$	28°	28°	$21^{\triangle}$	$19^{\triangle}$	$17^{\triangle}$	$15^{\triangle}$	$14^{\triangle}$	$13^{\triangle}$	$10^{\triangle}$	$9^{\triangle}$	$8^{\triangle}$	$7^{\triangle}$	6°	6°
29	$57^{\dagger}$	$43^{\dagger}$	$38^{\dagger}$	$36^{\dagger}$	29°	29°	$33^{\dagger}$	29°	29°	29°	29°	29°	29°	$30^{\triangle}$	29°	29°	29°
30	$59^{\dagger}$	$44^{\dagger}$	$39^{\dagger}$	36°	$25^{\triangle}$	$22^{\triangle}$	$20^{\triangle}$	$18^{\triangle}$	$17^{\triangle}$	$15^{\triangle}$	$14^{\triangle}$	$14^{\triangle}$	$9^{\triangle}$	$9^{\triangle}$	$8^{\triangle}$	$7^{\triangle}$	$6^{\triangle}$
31	$61^{\dagger}$	$46^{\dagger}$	$41^{\dagger}$	31°	$37^{\triangle\,\dagger}$	31°	31°	31°	31°	31°	31°	31°	31°	31°	$33^{\triangle\,\dagger}$	31°	31°
32	$63^{\dagger}$	$47^{\dagger}$	$42^{\dagger}$	$32^{\triangle}$	$32^{\triangle}$	$32^{\triangle}$	$33^{\triangle}$	$32^{\triangle}$	$32^{\triangle}$	$32^{\triangle}$	$32^{\triangle}$	$32^{\triangle}$	$32^{\triangle}$	$32^{\triangle}$	$33^{\triangle}$	$32^{\triangle}$	$32^{\triangle}$
33	$65^{\dagger}$	$49^{\dagger}$	$43^{\dagger}$	$41^{\dagger}$	$33^{\triangle}$	$25^{\triangle}$	$22^{\triangle}$	$20^{\triangle}$	$18^{\triangle}$	$16^{\triangle}$	$16^{\triangle}$	$16^{\triangle}$	$16^{\triangle}$	$16^{\triangle}$	$17^{\triangle}$	$6^{\triangle}$	$6^{\triangle}$
34	$67^{\dagger}$	$50^{\dagger}$	$45^{\dagger}$	$34^{\triangle}$	$28^{\triangle}$	$25^{\triangle}$	$23^{\triangle}$	$21^{\triangle}$	$19^{\triangle}$	$17^{\triangle}$	$17^{\triangle}$	$17^{\triangle}$	$17^{\triangle}$	$17^{\triangle}$	$17^{\triangle}$	$18^{\triangle}$	$5^{\triangle}$
35	$69^{\dagger}$	$52^{\dagger}$	45°	$42^{\triangle}$	$35^{\triangle}$	$36^{\triangle}$	$24^{\triangle}$	$22^{\triangle}$	$20^{\triangle}$	$16^{\triangle}$	$16^{\triangle}$	$16^{\triangle}$	$16^{\triangle}$	$16^{\triangle}$	$17^{\triangle}$	$7^{\triangle}$	$7^{\triangle}$

Blank: $p > g$, so $W(g,p) = 1$ (omitted). **Bold**: improves a cited lower bound; †: attains the upper bound; ∘: matches the comparison lower bound; △: literature comparison incomplete.

Table 5: Constructive lower bounds $C(g,p)$ for $p \le g$: $g = 19$–35, $p = 19$–35. Blank cells have $p > g$ and hence $W(g,p) = 1$.

$g\backslash p$	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35
19	20†																
20	5°	6°															
21	6°	6°	7°														
22	4°	4°	4°	5°													
23	23°	23°	23°	23°	24†												
24	8°	8°	8°	8°	8°	9°											
25	25°	25°	25°	25°	25°	25°	26†										
26	5°	5°	5°	5°	5°	5°	5°	6°									
27	27°	27°	27°	27°	27°	27°	27°	27°	28†								
28	6°	6°	6°	6°	6°	6°	6°	6°	6°	7°							
29	29°	29°	29°	29°	29°	29°	29°	29°	29°	29°	30†						
30	6$^{\triangle}$	6$^{\triangle}$	5°	5°	5°	5°	5°	5°	5°	5°	5°	6°					
31	31°	31°	31°	31°	31°	31°	31°	31°	31°	31°	31°	31°	32†				
32	32$^{\triangle}$	32$^{\triangle}$	32$^{\triangle}$	32$^{\triangle}$	32$^{\triangle}$	32$^{\triangle}$	32$^{\triangle}$	32$^{\triangle}$	32$^{\triangle}$	32$^{\triangle}$	32$^{\triangle}$	32$^{\triangle}$	32$^{\triangle}$	33†			
33	6$^{\triangle}$	6$^{\triangle}$	6$^{\triangle}$	6$^{\triangle}$	6$^{\triangle}$	6$^{\triangle}$	6$^{\triangle}$	6$^{\triangle}$	6$^{\triangle}$	6$^{\triangle}$	6$^{\triangle}$	6$^{\triangle}$	6$^{\triangle}$	6$^{\triangle}$	7°		
34	5$^{\triangle}$	5$^{\triangle}$	5$^{\triangle}$	5$^{\triangle}$	5$^{\triangle}$	5$^{\triangle}$	5$^{\triangle}$	5$^{\triangle}$	5$^{\triangle}$	5$^{\triangle}$	5$^{\triangle}$	5$^{\triangle}$	5$^{\triangle}$	5$^{\triangle}$	5$^{\triangle}$	6°	
35	7$^{\triangle}$	7$^{\triangle}$	7$^{\triangle}$	7$^{\triangle}$	7$^{\triangle}$	7$^{\triangle}$	7$^{\triangle}$	7$^{\triangle}$	7$^{\triangle}$	7$^{\triangle}$	7$^{\triangle}$	7$^{\triangle}$	7$^{\triangle}$	7$^{\triangle}$	7$^{\triangle}$	7$^{\triangle}$	8$^{\triangle}$

Blank: $p > g$, so $W(g,p) = 1$ (omitted). **Bold**: improves a cited lower bound; †: attains the upper bound; ∘: matches the comparison lower bound; △: literature comparison incomplete.

Table 6: Constructive lower bounds $C(g,p)$ for $p \le g$: $g = 36$–50, $p = 2$–18. Blank cells have $p > g$ and hence $W(g,p) = 1$.

$g\backslash p$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
36	$71^{\dagger}$	$53^{\dagger}$	$47^{\dagger}$	$36^{\triangle}$	$39^{\triangle}$	$36^{\triangle}$	$41^{\triangle\,\dagger}$	$37^{\triangle}$	$19^{\triangle}$	$17^{\triangle}$	$17^{\triangle}$	$17^{\triangle}$	$17^{\triangle}$	$17^{\triangle}$	$17^{\triangle}$	$18^{\triangle}$	$10^{\triangle}$
37	$73^{\dagger}$	$55^{\dagger}$	$49^{\dagger}$	$46^{\triangle\,\dagger}$	37°	37°	37°	37°	37°	37°	37°	37°	37°	37°	37°	37°	37°
38	$75^{\dagger}$	$56^{\dagger}$	$50^{\triangle\,\dagger}$	$38^{\triangle}$	$31^{\triangle}$	$28^{\triangle}$	$25^{\triangle}$	$24^{\triangle}$	$21^{\triangle}$	$19^{\triangle}$	$19^{\triangle}$	$19^{\triangle}$	$19^{\triangle}$	$19^{\triangle}$	$19^{\triangle}$	$19^{\triangle}$	$19^{\triangle}$
39	$77^{\dagger}$	$58^{\dagger}$	$51^{\dagger}$	39°	$39^{\triangle}$	$29^{\triangle}$	$26^{\triangle}$	$24^{\triangle}$	$21^{\triangle}$	$16^{\triangle}$	$16^{\triangle}$	$16^{\triangle}$	$16^{\triangle}$	$16^{\triangle}$	$17^{\triangle}$	$6^{\triangle}$	$6^{\triangle}$
40	$79^{\dagger}$	$59^{\dagger}$	$53^{\dagger}$	$48^{\triangle}$	$40^{\triangle}$	$40^{\triangle}$	$41^{\triangle}$	$25^{\triangle}$	$23^{\triangle}$	$17^{\triangle}$	$17^{\triangle}$	$17^{\triangle}$	$17^{\triangle}$	$17^{\triangle}$	$17^{\triangle}$	$18^{\triangle}$	$8^{\triangle}$
41	$81^{\dagger}$	$61^{\dagger}$	$53^{\triangle}$	$51^{\dagger}$	41°	41°	41°	41°	41°	41°	41°	41°	41°	41°	41°	41°	41°
42	$83^{\dagger}$	$62^{\dagger}$	$55^{\dagger}$	$42^{\triangle}$	$49^{\triangle}$	$30^{\triangle}$	$27^{\triangle}$	$26^{\triangle}$	$22^{\triangle}$	$19^{\triangle}$	$19^{\triangle}$	$19^{\triangle}$	$19^{\triangle}$	$19^{\triangle}$	$19^{\triangle}$	$19^{\triangle}$	$19^{\triangle}$
43	$85^{\dagger}$	$64^{\dagger}$	$57^{\triangle\,\dagger}$	43°	43°	43°	43°	43°	43°	43°	43°	43°	43°	43°	43°	43°	43°
44	$87^{\dagger}$	$65^{\dagger}$	$58^{\triangle\,\dagger}$	$44^{\triangle}$	$44^{\triangle}$	$31^{\triangle}$	$29^{\triangle}$	$25^{\triangle}$	$24^{\triangle}$	$19^{\triangle}$	$19^{\triangle}$	$19^{\triangle}$	$19^{\triangle}$	$19^{\triangle}$	$19^{\triangle}$	$19^{\triangle}$	$19^{\triangle}$
45	$89^{\dagger}$	$67^{\dagger}$	$59^{\dagger}$	$55^{\triangle}$	$45^{\triangle}$	45°	$29^{\triangle}$	$27^{\triangle}$	$24^{\triangle}$	$16^{\triangle}$	$16^{\triangle}$	$16^{\triangle}$	$16^{\triangle}$	$16^{\triangle}$	$17^{\triangle}$	$7^{\triangle}$	$7^{\triangle}$
46	$91^{\dagger}$	$68^{\dagger}$	$61^{\dagger}$	$46^{\triangle}$	$35^{\triangle}$	$32^{\triangle}$	$29^{\triangle}$	$27^{\triangle}$	$25^{\triangle}$	$23^{\triangle}$	$24^{\triangle}$	$23^{\triangle}$	$23^{\triangle}$	$23^{\triangle}$	$23^{\triangle}$	$23^{\triangle}$	$23^{\triangle}$
47	$93^{\dagger}$	$70^{\dagger}$	$61^{\triangle}$	47°	47°	47°	47°	47°	47°	47°	47°	47°	47°	47°	47°	47°	47°
48	$95^{\dagger}$	$71^{\dagger}$	$63^{\dagger}$	$48^{\triangle}$	$56^{\triangle}$	$48^{\triangle}$	$49^{\triangle}$	$48^{\triangle}$	$26^{\triangle}$	$23^{\triangle}$	$24^{\triangle}$	$23^{\triangle}$	$23^{\triangle}$	$23^{\triangle}$	$23^{\triangle}$	$23^{\triangle}$	$23^{\triangle}$
49	$97^{\dagger}$	$73^{\dagger}$	$65^{\triangle\,\dagger}$	$61^{\dagger}$	$49^{\triangle}$	$57^{\triangle\,\dagger}$	$49^{\triangle}$	$49^{\triangle}$	$49^{\triangle}$	$49^{\triangle}$	$49^{\triangle}$	$49^{\triangle}$	$49^{\triangle}$	$49^{\triangle}$	$49^{\triangle}$	$49^{\triangle}$	$49^{\triangle}$
50	$99^{\dagger}$	$74^{\dagger}$	$65^{\triangle}$	$60^{\triangle}$	$50^{\triangle}$	$50^{\triangle}$	$32^{\triangle}$	$29^{\triangle}$	$27^{\triangle}$	$25^{\triangle}$	$25^{\triangle}$	$26^{\triangle}$	$25^{\triangle}$	$25^{\triangle}$	$25^{\triangle}$	$25^{\triangle}$	$25^{\triangle}$

Blank: $p > g$, so $W(g,p) = 1$ (omitted). **Bold**: improves a cited lower bound; †: attains the upper bound; ○: matches the comparison lower bound; △: literature comparison incomplete.

Table 7: Constructive lower bounds $C(g,p)$ for $p \le g$: $g = 36$–50, $p = 19$–35. Blank cells have $p > g$ and hence $W(g,p) = 1$.

$g\backslash p$	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35
36	$9^{\triangle}$	$9^{\triangle}$	$9^{\triangle}$	$9^{\triangle}$	$9^{\triangle}$	$9^{\triangle}$	$9^{\triangle}$	$9^{\triangle}$	$9^{\triangle}$	$9^{\triangle}$	$9^{\triangle}$	$9^{\triangle}$	$9^{\triangle}$	$9^{\triangle}$	$9^{\triangle}$	$9^{\triangle}$	$9^{\triangle}$
37	$38^{\triangle}$	37°	37°	37°	37°	37°	37°	37°	37°	37°	37°	37°	37°	37°	37°	37°	37°
38	$20^{\triangle}$	$5^{\triangle}$	$5^{\triangle}$	$5^{\triangle}$	$5^{\triangle}$	$5^{\triangle}$	$5^{\triangle}$	$5^{\triangle}$	$5^{\triangle}$	$5^{\triangle}$	$5^{\triangle}$	$5^{\triangle}$	$5^{\triangle}$	$5^{\triangle}$	$5^{\triangle}$	$5^{\triangle}$	$5^{\triangle}$
39	$6^{\triangle}$	$6^{\triangle}$	$6^{\triangle}$	$6^{\triangle}$	$6^{\triangle}$	$6^{\triangle}$	$6^{\triangle}$	$6^{\triangle}$	$6^{\triangle}$	$6^{\triangle}$	$6^{\triangle}$	$6^{\triangle}$	$6^{\triangle}$	$6^{\triangle}$	$6^{\triangle}$	$6^{\triangle}$	$6^{\triangle}$
40	$8^{\triangle}$	$9^{\triangle}$	$8^{\triangle}$	$8^{\triangle}$	$8^{\triangle}$	$8^{\triangle}$	$8^{\triangle}$	$8^{\triangle}$	$8^{\triangle}$	$8^{\triangle}$	$8^{\triangle}$	$8^{\triangle}$	$8^{\triangle}$	$8^{\triangle}$	$8^{\triangle}$	$8^{\triangle}$	$8^{\triangle}$
41	41°	41°	$42^{\triangle}$	41°	41°	41°	41°	41°	41°	41°	41°	41°	41°	41°	41°	41°	41°
42	$20^{\triangle}$	$6^{\triangle}$	$7^{\triangle}$	$6^{\triangle}$	$6^{\triangle}$	$6^{\triangle}$	$6^{\triangle}$	$6^{\triangle}$	$6^{\triangle}$	$6^{\triangle}$	$6^{\triangle}$	$6^{\triangle}$	$6^{\triangle}$	$6^{\triangle}$	$6^{\triangle}$	$6^{\triangle}$	$6^{\triangle}$
43	43°	43°	43°	$44^{\triangle}$	43°	43°	43°	43°	43°	43°	43°	43°	43°	43°	43°	43°	43°
44	$20^{\triangle}$	$6^{\triangle}$	$7^{\triangle}$	$7^{\triangle}$	$6^{\triangle}$	$6^{\triangle}$	$6^{\triangle}$	$6^{\triangle}$	$6^{\triangle}$	$6^{\triangle}$	$6^{\triangle}$	$6^{\triangle}$	$6^{\triangle}$	$6^{\triangle}$	$6^{\triangle}$	$6^{\triangle}$	$6^{\triangle}$
45	$7^{\triangle}$	$7^{\triangle}$	$7^{\triangle}$	$7^{\triangle}$	$7^{\triangle}$	$7^{\triangle}$	$7^{\triangle}$	$7^{\triangle}$	$7^{\triangle}$	$7^{\triangle}$	$7^{\triangle}$	$7^{\triangle}$	$7^{\triangle}$	$7^{\triangle}$	$7^{\triangle}$	$7^{\triangle}$	$7^{\triangle}$
46	$23^{\triangle}$	$23^{\triangle}$	$23^{\triangle}$	$23^{\triangle}$	$24^{\triangle}$	$5^{\triangle}$	$5^{\triangle}$	$5^{\triangle}$	$5^{\triangle}$	$5^{\triangle}$	$5^{\triangle}$	$5^{\triangle}$	$5^{\triangle}$	$5^{\triangle}$	$5^{\triangle}$	$5^{\triangle}$	$5^{\triangle}$
47	47°	47°	47°	47°	47°	$48^{\triangle}$	47°	47°	47°	47°	47°	47°	47°	47°	47°	47°	47°
48	$23^{\triangle}$	$23^{\triangle}$	$23^{\triangle}$	$23^{\triangle}$	$24^{\triangle}$	$10^{\triangle}$	$9^{\triangle}$	$9^{\triangle}$	$9^{\triangle}$	$9^{\triangle}$	$9^{\triangle}$	$9^{\triangle}$	$9^{\triangle}$	$9^{\triangle}$	$9^{\triangle}$	$9^{\triangle}$	$9^{\triangle}$
49	$49^{\triangle}$	$49^{\triangle}$	$49^{\triangle}$	$49^{\triangle}$	$49^{\triangle}$	$49^{\triangle}$	$50^{\triangle}$	$49^{\triangle}$	$49^{\triangle}$	$49^{\triangle}$	$49^{\triangle}$	$49^{\triangle}$	$49^{\triangle}$	$49^{\triangle}$	$49^{\triangle}$	$49^{\triangle}$	$49^{\triangle}$
50	$25^{\triangle}$	$25^{\triangle}$	$25^{\triangle}$	$25^{\triangle}$	$25^{\triangle}$	$25^{\triangle}$	$26^{\triangle}$	$7^{\triangle}$	$7^{\triangle}$	$7^{\triangle}$	$7^{\triangle}$	$7^{\triangle}$	$7^{\triangle}$	$7^{\triangle}$	$7^{\triangle}$	$7^{\triangle}$	$7^{\triangle}$

Blank: $p > g$, so $W(g,p) = 1$ (omitted). **Bold**: improves a cited lower bound; †: attains the upper bound; ○: matches the comparison lower bound; △: literature comparison incomplete.

Table 8: Constructive lower bounds $C(g,p)$ for $p \le g$: $g = 36$–50, $p = 36$–50. Blank cells have $p > g$ and hence $W(g,p) = 1$.

$g\backslash p$	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
36	10°														
37	37°	$38^{\dagger}$													
38	$5^{\triangle}$	$5^{\triangle}$	6°												
39	$6^{\triangle}$	$6^{\triangle}$	$6^{\triangle}$	7°											
40	$8^{\triangle}$	$8^{\triangle}$	$8^{\triangle}$	$8^{\triangle}$	9°										
41	41°	41°	41°	41°	41°	$42^{\dagger}$									
42	$6^{\triangle}$	$6^{\triangle}$	$6^{\triangle}$	$6^{\triangle}$	$6^{\triangle}$	$6^{\triangle}$	7°								
43	43°	43°	43°	43°	43°	43°	43°	$44^{\dagger}$							
44	$6^{\triangle}$	$6^{\triangle}$	$6^{\triangle}$	$6^{\triangle}$	$6^{\triangle}$	$6^{\triangle}$	$6^{\triangle}$	$6^{\triangle}$	7°						
45	$7^{\triangle}$	$7^{\triangle}$	$7^{\triangle}$	$7^{\triangle}$	$7^{\triangle}$	$7^{\triangle}$	$7^{\triangle}$	$7^{\triangle}$	$7^{\triangle}$	8°					
46	$5^{\triangle}$	$5^{\triangle}$	$5^{\triangle}$	$5^{\triangle}$	$5^{\triangle}$	$5^{\triangle}$	$5^{\triangle}$	$5^{\triangle}$	$5^{\triangle}$	$5^{\triangle}$	6°				
47	47°	47°	47°	47°	47°	47°	47°	47°	47°	47°	47°	$48^{\dagger}$			
48	$9^{\triangle}$	$9^{\triangle}$	$9^{\triangle}$	$9^{\triangle}$	$9^{\triangle}$	$9^{\triangle}$	$9^{\triangle}$	$9^{\triangle}$	$9^{\triangle}$	$9^{\triangle}$	$9^{\triangle}$	$9^{\triangle}$	10°		
49	$49^{\triangle}$	$49^{\triangle}$	$49^{\triangle}$	$49^{\triangle}$	$49^{\triangle}$	$49^{\triangle}$	$49^{\triangle}$	$49^{\triangle}$	$49^{\triangle}$	$49^{\triangle}$	$49^{\triangle}$	$49^{\triangle}$	$49^{\triangle}$	$50^{\dagger}$	
50	$7^{\triangle}$	$7^{\triangle}$	$7^{\triangle}$	$7^{\triangle}$	$7^{\triangle}$	$7^{\triangle}$	$7^{\triangle}$	$7^{\triangle}$	$7^{\triangle}$	$7^{\triangle}$	$7^{\triangle}$	$7^{\triangle}$	$7^{\triangle}$	$7^{\triangle}$	8°

Blank: $p > g$, so $W(g,p) = 1$ (omitted). **Bold**: improves a cited lower bound; †: attains the upper bound; ∘: matches the comparison lower bound; △: literature comparison incomplete.

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